SOME ENERGIES OF REGULAR FUZZY GRAPHS

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Abstract. Energy of a fuzzy graph is defined as the sum of absolute values of the eigenvalues of the adjacency matrix of the fuzzy graph. Similarly, the distance (resistance distance) energy of a fuzzy graph G is defined as the sum of the absolute values of the eigenvalues of the distance (resistance distance) matrix of G. Also, the Laplacian energy of a graph G is equal to the sum of distances of the Laplacian eigenvalues of G and the average degree of G. In this paper, we compute mentioned energies for regular fuzzy graph which its crisp graph is a cycle.

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1. Introduction

Many real world systems can be modeled using graphs. Graphs represent the connections between the entities in these systems. The pictorial representation of a graph consists of a set [i]of points joined by arcs. To make use of computers to solve problems on graphs, they had to be stored in the memory of computers. This is done using matrices. Many kinds of matrices are associated with a graph. For instance, adjacency matrix, Laplacian matrix, distance matrix.

The spectrum of one such matrix, adjacency matrix is called the spectrum of the graph. The properties of the spectrum of a graph is related to the properties of the graph. The area of graph theory that deals with this is called spectral graph theory. The spectrum of a graph first appeared in a paper by Collatz and Sinogowitz in 1957. At present, it is widely studied owing to its applications in physics, chemistry, computer science and other branches of mathematics. Cvetkovic and Gutman have discussed these applications in detail in [4].

A concept related to the spectrum of a graph is that of energy. As its name suggests, it is inspired by energy in chemistry. The study of π-electron energy in chemistry dates back to 1940's but Gutman is defined it mathematically for all graphs. Organic molecules can be represented by graphs called molecular graphs. In case of unsaturated conjugated hydrocarbons, the energy of π- electrons of the molecule is approximately the energy of its molecular graph [3].

Fuzzy graphs are generalizations of graphs. Fuzzy graphs are encountered in fuzzy set theory. A fuzzy set was defined by Zadeh. Rosenfeld (1975) considered fuzzy relations on fuzzy sets [15] and developed the theory of fuzzy graphs, and then some basic fuzzy graph theoretic concepts and applications have been indicated, many authors found deeper results and fuzzy analogues of many other graph theoretic concepts. The energy of a fuzzy graph and some bounds on energy of fuzzy graphs are studied in [1].

2. Preliminaries

It is quite well known that graphs are simply models of relations. A graph is a convenient way of representing information involving relationship between objects. The objects are represented by vertices and relations by edges. When there is vagueness in the description of the objects or in its relationships or in both, it is natural that we need to design a "Fuzzy Graph Model". We know that a graph is a symmetric binary relation on a nonempty set *V*. Similarly, a fuzzy graph is a symmetric binary fuzzy relation on a fuzzy subset.

Let *V* be a nonempty set. A fuzzy subset of *V* is a function $\sigma: V \to [0,1]$. σ is called the membership function and $\sigma(v)$ is called the membership of *v* where $v \in V$.

Define a fuzzy subset $\mu: V \times V \to [0,1]$ as $\mu(u,v) \leq \min\{\sigma(u), \sigma(v)\}$. Then, μ is called a fuzzy relation on σ. $\mu(v_i, v_j)$ is interpreted as the strength of relation between v_i and v_j [11]. μ is said to be symmetric, if $\mu\left(v_i, v_j\right) = \mu(v_j, v_i)$ for $u, v \in V$.

A fuzzy relation can also be expressed by a matrix called fuzzy relation matrix $M = [m_{ij}]$ where $m_{ij} = \mu(v_i, v_j)$.

A fuzzy graph with *V* as the underlying set is a pair of functions $G = (\sigma, \mu)$, where σ is a fuzzy subset of a set *V* and μ is a fuzzy relation on σ [12]. The underlying crisp graph of $G = (\sigma, \mu)$ is denoted by $G^* = (V, E)$ where $E \subseteq V \times$.

Throughout this paper, we suppose $G = (\sigma, \mu)$ is undirect, without loops and $\sigma(v) = 1$, for each $v \in V$.

The adjacency matrix $A(G)$ of a fuzzy graph $G = (\sigma, \mu)$ is an $n \times n$ matrix defned as $A(G) = [a_{ij}]$, where $a_{ij} = \mu(v_i, v_j)$. The eigenvalues of $A(G)$ are called eigenvalues of *G*.

Definition 2. 1. ([1]) Let $G = (\sigma, \mu)$ be a fuzzy graph and *A* be its adjacency matrix. *Energy* of *G* is defined as the sum of absolute values of eigenvalues $\tau_1 \geq \tau_2 \geq \cdots \geq \tau_n$ of *A* and is denoted by *E*(*G*) as

$$E(G) = \sum_{i=1}^{n} |\tau_i|.$$

The degree of vertex u in $G = (\sigma, \mu)$ is defined as $d_G(u) = \sum_{uv \in E} \mu(u,v)$. Moreover, If $d_G(u) = k$ for all $u \in V$ (i.e. if each vertex has same degree *k*), then G is said to be a regular fuzzy graph of degree *k* or a *k*−regular fuzzy graph.

The matrix $L(G) = D(G) - A(G)$ is defined as fuzzy Laplacian matrix of $G = (\sigma, \mu)$, where $A(G)$ is the adjacency matrix and $D(G)$ is the degree matrix of G. The eigenvalues of fuzzy Laplacian matrix L are called Laplacian eigenvalues of G.

Definition 2. 2. ([13]) The Laplacian energy $L(G)$ of a graph G is equal to the sum of distance of the Laplacian eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of G and the average degree $d(G)$ of G and is denoted by $LE(G)$ as

$$LE(G) = \sum_{i=1}^{n} \left| \lambda_i - \frac{2\sum_{\leq i<j\leq n} m_{ij}}{n} \right|.$$

2.1. Distance energy of fuzzy graphs

In ordinary graphs, distance energy is defined as sum absolute eigenvalues of distance matrix, where elements of distance matrix are distance between vertices [9]. In here, we generalized distance energy for fuzzy graphs.

Let $G = (\sigma, \mu)$ be a fuzzy graph. A path P of length n is a sequence of distinct nodes $u_0, u_1, \cdots, u_n$ such that $\mu(u_{i-1}, u_n) > 0,\ i = 1,2, \cdots, n$.

The μ- distance $d_\mu(u, v)$ is the smallest μ -length of any u - v path, where the μ-length of a path $P: u_0, u_1, \cdots, u_n$ is [15]

$$\ell(P) = \sum_{i=1}^{n} \frac{1}{\mu(u_{i-1}, u_i)}.$$

If n = 0, then define $\ell(P)$=0.

Definition 2.3. The μ-distance matrix of a fuzzy graph G which are said the fuzzy μ-distance matrix, is defined as a square matrix $D_\mu(G) = [d_{ij}]$, where d_{ij} is the μ-distance between the vertices v_i and v_j in G.

Example 2.4. The μ-distance matrix of a fuzzy graph G_1 (Fig. 1) is

$$D_\mu(G) = \begin{pmatrix} 0 & 3.3333 & 8.3333 & 10 \\ 3.3333 & 0 & 5 & 13.3333 \\ 8.3333 & 5 & 0 & 10 \\ 10 & 13.3333 & 10 & 0 \end{pmatrix}.$$

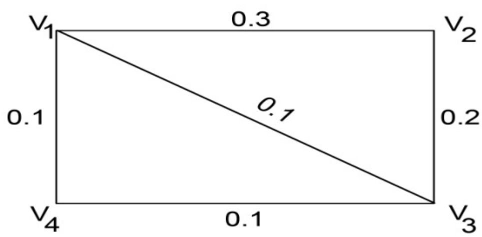

Fig. 1. A fuzzy graph G_1

The eigenvalues of the fuzzy μ-distance matrix $D_\mu(G)$ are denoted by $\delta_1, \delta_2, \cdots, \delta_n$ and are said to be the D_μ-eigenvalues of G and to form the D_μ-spectrum of G, denoted by $Spec_{D_\mu}(G)$. Since the fuzzy μ-distance matrix is symmetric, its eigenvalues are real and can be ordered as $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$. We have

$$\sum_{i=1}^{n} \delta_i = trace\big(D(G)\big) = \sum_{i=1}^{n} d_{ii} = 0,$$

$$\sum_{i=1}^{n} {\delta_i}^2 = trace\big[D_\mu(G)\big]^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\big(d_{ij}\big)^2 = 2\sum_{1\leq i<j\leq n}\big(d_{ij}\big)^2.$$

Definition 2.5. The μ-distance energy $D\mu E(G)$ (abbreviation, $DE(G)$) of a fuzzy graph G is defined as

$$D\mu E(G) = \sum_{i=1}^{n} |\delta_i|.$$

For example, in fuzzy graph G_1 (Fig. 1),

$$Spec_{D_\mu}(G_1) = \{25.5953, -1.7799, -8.3871, -15.4283\}$$

so, $D\mu E(G_1) = 51.1907$.

Lemma 2.6. ([7]) The distance matrices of any connected graph on *n* vertices, $n \geq 2$, has exactly one positive eigenvalue and exactly *n*-1 negative eigenvalues.

The above lemma is true for fuzzy graphs. So we have following corollary.

Corollary 2.7. Let $D_\mu(G)$ be μ-distance matrix of fuzzy graph $G = (\sigma, \mu)$ with eigenvalues $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$. Then $D_\mu E(G) = 2\delta_1$**.**

In following theorem, we give bonds for $D_\mu E(G)$ which its proofs are fully analogous to what Ramane et al. in [14] has done in the case of the ordinary graph. Hence proofs are omitted.

Theorem 2.8. Let $G = (\sigma, \mu)$ be a fuzzy graph with *n* vertices. Then

a) $\sqrt{2\sum_{1\leq i<j\leq n}\big(d_{ij}\big)^2} \leq D_\mu E(G) \leq \sqrt{2n\sum_{1\leq i<j\leq n}\big(d_{ij}\big)^2}$.

b) $\sqrt{2\sum_{1\leq i<j\leq n}\big(d_{ij}\big)^2 + n(n-1)|\det(D(G)|^{\frac{2}{n}}} \leq D_\mu E(G)$.

c) $\sqrt{2(n-1)\sum_{1\leq i<j\leq n}\big(d_{ij}\big)^2 + n|\det(D(G)|^{\frac{2}{n}}} \geq D_\mu E(G)$

d) $D_\mu E(G) \leq \frac{2}{n}\sum_{1\leq i<j\leq n}\left(d_{ij}\right)^2 + \sqrt{(n-1)\left[2\sum_{1\leq i<j\leq n}\left(d_{ij}\right)^2 - \left(\frac{2}{n}\sum_{1\leq i<j\leq n}\left(d_{ij}\right)^2\right)^2\right]}$

2.2. Resistance distance energy of fuzzy graphs

The concept of resistance distance, introduced by Klein and Randic' [10], arises naturally from several different considerations and is also mathematically more attractive than the classical distance. For more background information about resistance distance we refer to [2, 6, 7, 16].

In here, we generalized some definitions of resistance distance for fuzzy graphs.

Definition 2.9. Let $L(G)$ be the fuzzy Laplacian matrix of G. Let $L(i)$ be the matrix resulting from removing the ith row and column of the fuzzy Laplacian L and let $L(i,j)$ the matrix resulting from removing both the ith and jth rows and columns of L. The resistance distance r_{ij} between v_i and v_j is zero if $i = j$, and if $i \neq j$, then $r_{ij} = \frac{\det(L(i,j))}{\det(L(i))}$.

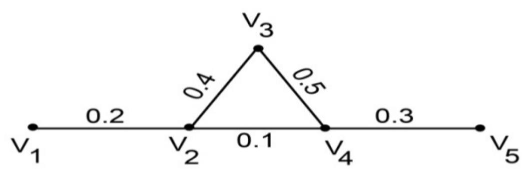

Fig. 2. A fuzzy graph G_2

For example, in fuzzy graph G_2 (Fig. 2), we have

$$L(G_2) = \begin{pmatrix} 0.2 & -0.2 & 0 & 0 & 0 \\ -0.2 & 0.7 & -0.4 & -0.1 & 0 \\ 0 & -0.4 & 0.9 & -0.5 & 0 \\ 0 & -0.1 & -0.5 & 0.9 & -0.3 \\ 0 & 0 & 0 & -0.3 & 0.3 \end{pmatrix},$$

$$L(1) = \begin{pmatrix} 0.7 & -0.4 & -0.1 & 0 \\ -0.4 & 0.9 & -0.5 & 0 \\ -0.1 & -0.5 & 0.9 & -0.3 \\ 0 & 0 & -0.3 & 0.3 \end{pmatrix},$$

$$L(1,2) = \begin{pmatrix} 0.9 & -0.5 & 0 \\ -0.5 & 0.9 & -0.3 \\ 0 & -0.3 & 0.3 \end{pmatrix},\ L(1,5) = \begin{pmatrix} 0.7 & -0.4 & -0.1 \\ -0.4 & 0.9 & -0.5 \\ -0.1 & -0.5 & 0.9 \end{pmatrix}.$$

Thus $r_{12} = \frac{\det(L(1,2))}{\det(L(1))} = \frac{0.0870}{0.0174} = 5$ and $r_{15} = \frac{\det(L(1,5))}{\det(L(1))} = \frac{0.1990}{0.0174} = 11.4368.$

Definition 2.10. The second definition is in terms of electrical networks. Think of G as an electrical network in which a unit resistance is placed along each edge. Current is allowed to enter the network only at vertex v_i and leave the network only at vertex v_j. Then the resistance distance between v_i and v_j is the "effective resistance" between v_i and v_j.

For example, by this definition in fuzzy graph G_2 (Fig. 2) we obtain,

$$r_{12} = \frac{1}{0.2} = 5, \quad r_{15} = \frac{1}{0.2} + \frac{1}{\frac{1}{\frac{1}{0.1}} + \frac{1}{\frac{1}{0.4}+\frac{1}{0.5}}} + \frac{1}{0.3} = \frac{1}{0.2667} = 11.4368.$$

Definition 2.11. Let J denote the square matrix of order n such that all of whose elements are unity and L is the fuzzy Laplacian matrix of G. Then resistance distance between v_i and v_j is $r_{ij} = x_{ii} + x_{jj} - 2x_{ij}$, where $X = (x_{ij}) = \left(L + \frac{1}{n}J\right)^{-1}$.

For example, in fuzzy graph G_2 (Fig. 2), we have

$$X = \begin{pmatrix} 4.3885 & 0.3885 & -0.4179 & -1.1977 & -1.8644 \\ 0.3885 & 1.3885 & 0.2851 & -0.1977 & -0.8644 \\ -0.4179 & 0.2851 & 1.2506 & 0.4230 & -0.2437 \\ -1.1977 & -0.1977 & 0.4230 & 1.3195 & 0.6529 \\ -1.8644 & -0.8644 & -0.2437 & 0.6529 & 3.3195 \end{pmatrix},$$

So

$$r_{12} = x_{11} + x_{22} - 2x_{12} = 4.3885 + 1.3885 - 2 * 0.3885 = 5,$$

$$r_{15} = x_{11} + x_{55} - 2x_{15} = 4.3885 + 3.3195 + 2 * 1.8644 = 11.4368.$$

If there is a unique $u - v$ path in G, then it is clear from definition 2.10 that the resistance distance and μ-distance between u and v coincide. Thus, we recall resistance distance r_{ij} with μ-resistance distance.

Definition 2.12. let $G = (\sigma, \mu)$ be a fuzzy graph. The matrix whose (i, j)-entry is r_{ij}, is called the fuzzy μ-resistance distance matrix and will be denoted by $RD_\mu(G)$. This matrix is symmetric and has a zero diagonal.

Example 2.13. The μ-resistance distance matrix of a fuzzy graph G_1 (Fig. 1) is

$$RD_\mu(G_1) = \begin{pmatrix} 0 & 2.5926 & 3.7037 & 5.9259 \\ 2.5926 & 0 & 3.3332 & 7.0369 \\ 3.7037 & 3.3332 & 0 & 5.9259 \\ 5.9259 & 7.0369 & 5.9259 & 0 \end{pmatrix}.$$

Let $\rho_1, \rho_2, \cdots, \rho_n$be the eigenvalues of the fuzzy μ-resistance-distance matrix $RD_\mu(G)$. Since this matrix is symmetric, its eigenvalues are real and can be ordered as $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$. Moreover, it is easy to prove that

$$\sum_{i=1}^{n} \rho_i = 0,$$

$$\sum_{i=1}^{n} {\rho_i}^2 = 2 \sum_{1 \leq i < j \leq n} \left(r_{ij}\right)^2.$$

Definition 2.14. The μ-resistance distance energy $ERD_\mu(G)$ of a fuzzy graph G is defined as sum of absolute values of the eigenvalues of the fuzzy μ –resistance distance matrix $RD_\mu(G)$,

$$RD_\mu E(G) = \sum_{i=1}^{n} |\rho_i|.$$

For example, in fuzzy graph G_1 (Fig. 1),

$$Spec_{RD_\mu}(G_1) = \{14.5852, -2.4434, -3.8138, -8.3279\},$$

so, $RD_\mu E(G_1) = 29.1703$.

Lemma 2.15. ([17]) The resistance distance matrices of any connected graph on n vertices, $n \geq 2$, have exactly one positive eigenvalue and exactly n-1 negative eigenvalues.

The above lemma is true for fuzzy graphs. So we have following corollary.

Corollary 2.16. Let $G = (\sigma, \mu)$ be a fuzzy graph and $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ be eigenvalues of $RD_\mu(G)$. Then $RD_\mu E(G) = 2\rho_1$.

In following theorem, we give bonds for $ERD_\mu(G)$ which its proofs are fully analogous to what Das *et al*. in [5] has done in the case of the ordinary graph. Hence proofs are omitted.

Theorem 2.17. Let $G = (\sigma, \mu)$ be a fuzzy graph with n vertices. Then

a) $\sqrt{2\sum_{1 \leq i<j \leq n} \left(r_{ij}\right)^2} \leq RD_\mu E(G) \leq \sqrt{2n\sum_{1 \leq i<j \leq n} \left(r_{ij}\right)^2}$.

b) $\sqrt{2\sum_{1 \leq i<j \leq n} \left(r_{ij}\right)^2 + n(n-1)\left|det(RD_\mu(G)\right|^{\frac{2}{n}}} \leq RD_\mu E(G)$.

c) $\sqrt{2(n-1)\sum_{1 \leq i<j \leq n} \left(r_{ij}\right)^2 + n\left|det(RD_\mu(G)\right|^{\frac{2}{n}}} \geq RD_\mu E(G)$

d) $RD_{\mu}E(G) \leq \frac{2}{n}\sum_{1\leq i<j\leq n}\left(r_{ij}\right)^2 + \sqrt{(n-1)\left[2\sum_{1\leq i<j\leq n}\left(r_{ij}\right)^2 - \left(\frac{2}{n}\sum_{1\leq i<j\leq n}\left(r_{ij}\right)^2\right)^2\right]}$

3. Result

In here, we compute the mentioned energies in pervious sections for regular fuzzy graph which its crisp graph is cycles.

Lemma 3.1. Let *G* be a *d*-regular fuzzy graph. Then $E(G) = LE(G)$.

Proof. Let G be *d*-regular. Also, Suppose $\tau_1 \geq \tau_2 \geq \cdots \geq \tau_n$ are eigenvalue of G and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are Laplacian eigenvalue of G. We have $\lambda_i = d - \tau_{n-i+1}$. So

$$\left|\lambda_i - \frac{2\sum_{\leq i<j\leq n} m_{ij}}{n}\right| = |\lambda_i - d| = |\tau_{n-i+1}|,$$

Therefore $E(G) = LE(G)$.

The following lemma is essential to continue.

Lemma 3.2. ([8]) Let $G = (\sigma, \mu)$ be a fuzzy graph where $G^* = (V, E)$ is an cycle of length *n*. Then

a) If *n* be odd, then *G* is regular iff μ is a constant function.
b) If *n* be even, then *G* is regular iff either μ is a constant function or alternate edges have same membership values.

Theorem 3. 3. Let $G = (\sigma, \mu)$ be a *d*-regular fuzzy graph where $G^* = (V, E)$ is an odd cycle of length *n*. Then

$$E(G) = d + 2d\sum_{k=1}^{\frac{n-1}{2}}\left|\cos\frac{2k\pi}{n}\right|.$$

Proof. According Lemma 3.2, μ is constant function. Suppose $\mu = c$. Then adjacency matrix A of G *is* cA^* where A^* is adjacency matrix of G^*. so eigenvalues λ_i of A are $c\lambda_i^*$ where λ_i^* are eigenvalues of G^*. we have Spec(G)={$2c\cos\frac{2k\pi}{n} | k = 0,\ 1, \cdots, n-1$}. Thus

$$E(G) = 2c\sum_{k=0}^{n-1}\left|\cos\frac{2k\pi}{n}\right|.$$

On other hand, $d = 2c$ and $\cos\frac{2k\pi}{n} = \cos\frac{2k(n-1)\pi}{n}$, for $k = 1, 2, \cdots, n-1$, and this completes the proof.

Theorem 3. 4. Let $G = (\sigma, \mu)$ be a d-regular fuzzy graph where $G^* = (V, E)$ is an even cycle of length n. Then $E(G) = 2d\sum_{k=0}^{\frac{n}{2}-1}\left|\cos\frac{2k\pi}{n}\right|$ if μ is a constant function, Otherwise

$$E(G) = 2\sum_{k=0}^{\frac{n}{2}-1}\sqrt{d^2 - 2c_1c_2(1 - \cos\frac{4k\pi}{n})}\,,$$

where, c_1, c_2 are membership values of edges.

Proof. According Lemma 3.2, we have two cases: a) either μ is a constant function or b) alternate edges have same membership values. In case (a), similarly theorem 3.2, following the result.

In case (b), by Lemma 3.2 we can let $\mu = c_1, c_2$. Since n is even, so G determines a bipartition of the vertex set into two nonempty parts V_1 and V_2 of sizes n, such that there are no walks of length 2 between V_1 and V_2. Let B is the representation matrix of the bipartite graph G. The graph G can be specified by specifying the matrix B. The adjacency matrix A of G can be write of the form $A = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$. The matrix A satisfies $A^2 = \begin{bmatrix} BB^T & 0 \\ 0 & B^TB \end{bmatrix}$ and so the eigenvalues of A^2 are those of BB^T together with those of B^TB. We have $B^TB = ({c_1}^2 + {c_2}^2)I + c_1c_2Q$, where I is $\frac{n}{2} \times \frac{n}{2}$-identity matrix and Q is the adjacency matrix of a simple cycle of length $\frac{n}{2}$. By Lemma 3. 1,

$$Spec(B^TB) = \left\{({c_1}^2 + {c_2}^2) + 2c_1c_2\cos\frac{4k\pi}{n}\middle|k = 0,1,2,\cdots,\frac{n}{2} - 1\right\}.$$

The eigenvalues of A are therefore square roots of these numbers, and by the symmetry of the spectrum of A, the eigenvalues of A are n numbers $\pm\sqrt{({c_1}^2 + {c_2}^2) + 2c_1c_2\cos\frac{4k\pi}{n}}$ for $k = 0,1,2,\cdots,\frac{n}{2} - 1$. We have $d = c_1 + c_2$ and this completes the proof.

Theorem 3.5. Let $G = (\sigma, \mu)$ be a d-regular fuzzy graph where $G^* = (V, E)$ is an cycle of length n. If μ is a constant function. Then

$$D_\mu E(G) = \begin{cases} \dfrac{n^2 - 1}{d}, & n \text{ is odd} \\ \dfrac{n^2}{d}, & n \text{ is even} \end{cases}.$$

Proof. Let $D_\mu(G) = [d_{ij}]$ be distance matrix of graph G with eigenvalues $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$. According Corollary 2.7, $D_\mu E(G) = 2\delta_1$. So it is sufficient to find the maximum eigenvalues δ_1.

Since G is regular, so by lemma 5.5, $\delta_1 = \sum_{j=1}^{n} d_{1j}$. Let D_1 be first row of matrix $D_\mu(G)$. By Lemmas 3.2, we can let μ=c. Thus for n odd we have

$$D_1 = \begin{bmatrix} 0 & \frac{1}{c} & 2\left(\frac{1}{c}\right) & 3\left(\frac{1}{c}\right) & \cdots & \left(\frac{n-1}{2}\right)\left(\frac{1}{c}\right) & \left(\frac{n-1}{2}\right)\left(\frac{1}{c}\right) & \left(\frac{n-3}{2}\right)\left(\frac{1}{c}\right) & \cdots & 2\left(\frac{1}{c}\right) & \frac{1}{c} \end{bmatrix},$$

and for n even we have

$$D_1 = \begin{bmatrix} 0 & \frac{1}{c} & 2\left(\frac{1}{c}\right) & 3\left(\frac{1}{c}\right) & \cdots & \left(\frac{n-2}{2}\right)\left(\frac{1}{c}\right) & \left(\frac{n}{2}\right)\left(\frac{1}{c}\right) & \left(\frac{n-2}{2}\right)\left(\frac{1}{c}\right) & \cdots & 2\left(\frac{1}{c}\right) & \frac{1}{c} \end{bmatrix}.$$

Therefore if n is odd, then

$$\sum_{j=1}^{n} d_{1j} = 2\left(\frac{1}{c}\right)\binom{\frac{n+1}{2}}{2} = \frac{n^2-1}{4c}.$$

and if n is even, then

$$\sum_{j=1}^{n} d_{1j} = 2\left(\frac{1}{c}\right)\binom{\frac{n}{2}}{2} + \left(\frac{n}{2}\right)\left(\frac{1}{c}\right) = \frac{n^2}{4c}.$$

Since d=2c, the result follow.

Theorem 3.6. Let $G = (\sigma, \mu)$ be a d-regular fuzzy graph where $G^* = (V, E)$ is an even cycle of length n. If μ is non-constant function. Then

$$D_\mu E(G) = \begin{cases} \dfrac{8kd}{c_1 c_2}, & n = 4k \\ \dfrac{8kd}{c_1 c_2} + \dfrac{2}{c_2}, & n = 4k+2, \quad c_1 < c_2 \end{cases},$$

where c_1 and c_2 are membership values of edges.

Proof. Let $D_\mu(G) = [d_{ij}]$ be distance matrix of graph G with eigenvalues $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$. According Lemma 3.2(b), alternate edges have same membership values. So, we can suppose $\mu = c_1, c_2$.

By Corollary 2.7, $ED_\mu(G) = 2\delta_1$. Now to calculate δ_1, as proof of pervious theorem we act. Therefor we obtain

$$\delta_1 = \begin{cases} 4k\left(\dfrac{1}{c_1} + \dfrac{1}{c_2}\right), & n = 4k \\ \dfrac{4k}{c_1} + \dfrac{4k+1}{c_2}, & n = 4k+2, \quad c_1 < c_2 \end{cases},$$

Since $d = c_1 + c_2$, the result follow.

Lemma 3.7. ([19-20]) Let $B = (B_{ij})$ be an $n \times n$ nonnegative, irreducible, symmetric matrix ($n \geq 2$) with row sums $B_1, B_2, \cdots, B_n$. If $\lambda_1(B)$ is the maximum eigenvalue of B, then

$$\sqrt{\frac{\sum_{i=1}^{n} B_i^2}{n}} \leq \lambda_1(B) \leq \max_{1\leq j\leq n} \sum_{i=1}^{n} B_{ij} \sqrt{\frac{B_j}{B_i}},$$

with equality holding if and only if $B_1 = B_2 = \cdots = B_n$ or if there is a permutation matrix Q such that $Q^T BQ = \begin{pmatrix} 0 & C \\ C^T & 0 \end{pmatrix}$, where all the row sums of C are equal.

Theorem 3.8. Let $G = (\sigma, \mu)$ be a d-regular fuzzy graph where $G^* = (V, E)$ is an cycle of length n. Also, suppose $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n = 0$ are eigenvalues of its fuzzy Laplacian graph. Then

$$RD_\mu E(G) = \sum_{i=1}^{n-1} \frac{1}{\lambda_i}.$$

Proof. By definition of resistance distance we have

$$\sum_{j=1}^{n}\sum_{i=1}^{n} r_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{n} x_{ii} + \sum_{j=1}^{n}\sum_{i=1}^{n} x_{jj} - 2\sum_{j=1}^{n}\sum_{i=1}^{n} x_{ij} =$$

$$ntrac(X) + ntrac(X) - 2n = 2n(trace(X) - 1) = 2n\sum_{i=1}^{n-1} \frac{1}{\lambda_i}.$$

On other hands, if $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ be eigenvalues of $RD_\mu(G)$, then by Lemma 3.7 we have

$$\rho_1 = \sum_{j=1}^{n} r_{1j} = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij} = 2\sum_{i=1}^{n-1} \frac{1}{\lambda_i}.$$

Since $RD_\mu E(G) = 2\rho_1$, so the proof is completed.

Finally, it is straightforward to realize that for graphs that contain no cycles, both matrices the fuzzy μ-distance and fuzzy μ-resistance distance coincide. However, the presence of cycles reduces the μ-resistance distance in comparison with the μ-distance. So, we have following theorem.

Theorem 3.9. Let $G = (\sigma, \mu)$ be a fuzzy graph. Then $RD_\mu E(G) \leq D_\mu E(G)$ with equality iff $G^* = (V, E)$ is acyclic.

Das *et al*. in [5] shown for any n-vertex tree T,

$$\det\big(D(G)\big) = \det\big(RD(G)\big) = (-1)^{n-1}(n-1)2^{n-2}.$$

With same argument for fuzzy graph $G = (\sigma, \mu)$ which its crisp graph is acyclic, we obtain

$$\det\left(D_\mu(G)\right) = \det\left(RD_\mu(G)\right) = (-1)^{n-1}2^{n-2}\left(\sum_{1\leq i<j\leq n} \frac{1}{m_{ij}}\right)\left(\prod_{1\leq i<j\leq n} \frac{1}{m_{ij}}\right).$$

Now, by theorems 2.8 and 2.17, we have the following result.

Corollary 3.10. Let $G = (\sigma, \mu)$ be a fuzzy graph with *n* vertices, where $G^* = (V, E)$ is acyclic. Then $\sqrt{2\sum_{1\le i<j\le n}\left(d_{ij}\right)^2 + 4n(n-1)\left[\frac{SP}{4}\right]^{\frac{2}{n}}} \le RD_{\mu}E(G) = D_{\mu}E(G) \le \sqrt{2(n-1)\sum_{1\le i<j\le n}\left(d_{ij}\right)^2 + 4n\left[\frac{SP}{4}\right]^{\frac{2}{n}}}$
where $S = \sum_{1\le i<j\le n}\frac{1}{m_{ij}}$ and $P = \prod_{1\le i<j\le n}\frac{1}{m_{ij}}$.

4. Conclusion

Energy and Laplacian energy of regular fuzzy graph with its crisp graph cycle are computed. Distance energy and resistance-distance energy for a fuzzy graph are defined. Some results on energy bounds for simple graphs are extended to fuzzy graphs. In fuzzy graph there are some another metrics, so In future studies can be discussed on energy of these distances. Further study on these energies of fuzzy graphs may reveal more analogous results of these kind and will be discussed in the forthcoming papers.

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Некоторые энергии регулярных нечетких графов

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РЕЗЮМЕ

Энергия нечеткого графа определяется как сумма абсолютных значений собственных значений матрицы смежности нечеткого графа. Точно так же энергия расстояния (сопротивления) нечеткого графа G определяется как сумма абсолютных значений собственных значений матрицы расстояния (расстояния сопротивления) G. Кроме того, энергия Лапласа графа G равна сумма расстояний собственных значений

Лапласа G и средняя степень G. В этой статье мы вычисляем упомянутые энергии для регулярного нечеткого графа, который его четким графом является циклом.
Ключевые слова: Матрица расстояний, энергетика, нечеткий график, расстояние сопротивления, спектр.